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Reliability equivalence factors for a series-parallel system assuming an exponentiated Weibull distribution

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Abstract. In this study, we discuss the reliability equivalence factors of a series-parallel system with n series subsystems connected in parallel where subsystem i has m_i independent and identically distributed components. We assume that each component has an exponentiated Weibull distribution. We consider improving the reliability of the system by (a) a reduction method and (b) several duplication methods: (i) hot duplication; (ii) cold duplication with perfect switch; (iii) cold duplication with imperfect switch. We compute two types of reliability equivalence factors: survival equivalence factors (SREF) and mean equivalence factors (MREF). We illustrate the results obtained with an application example.

1. Introduction

The concept of reliability equivalence factors was introduced by Rade in 1989 and published by him in Rade (1993a,b). He applied this concept to simple systems that consist of one component or two components connected in series or parallel. Later, Sarhan (2000, 2005) and Sarhan et al. (2008) applied this concept to more general systems. Most of the designs considered have components with exponential lifetime distributions although some studies applied this concept to other lifetime distributions, such as the Weibull distribution (El-Damcese, 2009) and gamma distribution (Xia and Zhang, 2007).

There are two main methods for improving a system's design. The first method is reduction, which involves improving the reliability of the system by reducing the failure rate by a factor ρ for some of the system components, where $\rho \in (0,1)$. This can be achieved by replacing standard components with more expensive, higher quality components. The second method for improving a system's design is redundancy duplication, which involves adding extra components in parallel to existing system components. There are three ways to add extra components to the system: hot duplication; cold duplication with perfect switch; cold duplication with imperfect switch. Sometimes, and for many different reasons such as high cost and space limitation, it is impossible to improve the reliability of the system by the redundancy duplication method. Reliability equivalence factors refer to the factors by which the failure rates of some of the system's components must be reduced in order to reach equality of the system reliability with that of a better system.

In this study, we consider a series-parallel system and assume that all the system's components are independent and follow the exponentiated Weibull distribution with identical parameters (Mudholkar and Srivastava, 1993). Firstly, we compute the reliability function and the mean time to failure (MTTF) of the original system. Secondly, we compute the reliability functions and MTTFs of the systems following improvement according to reduction, hot duplication and cold duplication (perfect and imperfect) methods. Thirdly, we equate the reliability function and the MTTF of the system improved according to the reduction method with the reliability function and the MTTF of the system improved according to each of the duplication methods to determine the reliability equivalence factors. Finally, we illustrate the results obtained with an application example by presenting table and figures.

2. Series-parallel system

The system we consider here is shown in Figure 1 and consists of n subsystems connected in parallel, where subsystem i consists of m_i components that are connected in series for $i = 1,2,\dots,n$. Such a system is usually referred to as a series-parallel system (El-Damcese, 2009), though some authors refer to it as a parallel-series system.

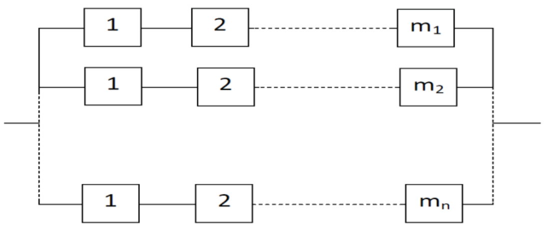

Figure 1 Series-parallel system

We assume that the lifetimes of all the system's components are independent and follow the exponentiated Weibull distribution with identical parameters (Mudholkar and Srivastava, 1993; Lai, 2014). The exponentiated Weibull distribution generalizes well known lifetime distributions including Weibull with two parameters, Rayleigh and generalized Rayleigh distributions.

Let $r_{ij}(t)$ be the reliability function of component j $(j = 1,2,\dots,m_i)$ in subsystem i $(i = 1,2,\dots,n)$ and let $R_i(t)$ be the reliability function of subsystem i. The above assumption implies that $r_{ij}(t) = r(t)$ where

$$r(t) = 1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta \tag{1}$$

for $t \geq 0$, as the lifetimes of components are unaffected by failures of other components. The reliability function of subsystem *i* then takes the form

$$R_i(t) = \prod_{j=1}^{m_i}\left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\} = \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i} \tag{2}$$

for $t \geq 0$, so the reliability function of the series-parallel system is

$$R(t) = 1 - \prod_{i=1}^{n}\{1 - R_i(t)\} = 1 - \prod_{i=1}^{n}\left[1 - \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i}\right] \tag{3}$$

for $t \geq 0$, and the mean time to failure of the series-parallel system is given by

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty \left(1 - \prod_{i=1}^{n}\left[1 - \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i}\right]\right)dt \tag{4}$$

3. Designs of improved systems

The two main approaches for improving a system are reduction methods and standby redundancy (duplication) methods. The latter comprise two variations, hot duplication and cold duplication. Furthermore, cold duplication can be performed with perfect switch or imperfect switch. In this section, we derive the reliability function and the mean time to failure for series-parallel systems improved according to the methods identified above.

3.1 *Reduction method*

As mentioned in the introduction, the reliability of a system can be improved by reducing the failure rate for some of the system's components by a factor $\rho \in (0,1)$. For the exponentiated Weibull distribution, reducing only the scale parameter reduces the failure rate. Here, we consider reducing a set A of the system's components by a factor ρ, in order to reduce the failure rate (hazard function) for the whole system. This is a logical procedure for the exponentiated Weibull distribution.

Define $a_i (i = 1,2,\dots,n)$ to be the number of components in subsystem *i* whose failure rate is reduced, so $a_i \in \{0,1,\dots,m_i\}$ and the cardinality of the set of improved components is $|A| = \sum_{i=1}^{n} a_i$. By comparison with Equation (2), we see that the reliability function $R_i^{(A)}(t)$ of subsystem *i* is then given by

$$R_i^{(A)}(t) = \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{a_i}\left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - a_i} \tag{5}$$

for $t \geq 0$ from Equation (1) and by comparison with Equation (3), since the components are connected in series. Then the reliability function of the system takes the form

$$R^{(A)}(t) = 1 - \prod_{i=1}^{n}\left[1 - \left\{1 - \left(1 - e^{-\rho\alpha t^\beta}\right)^\theta\right\}^{a_i}\left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - a_i}\right] \tag{6}$$

since the subsystems are connected in parallel. We can then compute the mean time to failure of this series-parallel system as

$$MTTF^{(A)} = \int_0^\infty \left(1 - \prod_{i=1}^{n}\left[1 - \left\{1 - \left(1 - e^{-\rho\alpha t^\beta}\right)^\theta\right\}^{a_i}\left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - a_i}\right]\right)dt \tag{7}$$

3.2 *Duplication methods*

Now we obtain the corresponding reliability measures of the system when it is improved by duplication. We derive the reliability function and the mean time to failure of the series-parallel system improved according to the hot duplication method and the cold duplication methods with perfect and imperfect switches.

3.2.1 *Hot duplication method*

This means that some of the system components are duplicated in parallel by similar components. We assume that in the hot duplication method each component of the set B is augmented by introducing a new but identical component in the same subsystem.

Let $b_i (i = 1,2, \dots, n)$ be the number of components in subsystem i whose reliability is improved according to the hot duplication method, so $b_i \in \{0,1, \dots, m_i\}$ and $|B| = \sum_{i=1}^{n} b_i$. The reliability function $R_i^{(B)}(t)$ of subsystem i is given by

$$R_i^{(B)}(t) = \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{2\theta}\right\}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - b_i} \tag{8}$$

since the components are connected in series. Then the reliability function of the system takes the form

$$R^{(B)}(t) = 1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{2\theta}\right\}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - b_i}\right] \tag{9}$$

for $t \geq 0$, and the mean time to failure of this series-parallel can then computed as

$$MTTF^{(B)} = \int_0^{\infty} \left(1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{2\theta}\right\}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - b_i}\right]\right) dt \tag{10}$$

3.2.2 *Cold duplication method with perfect switch*

This approach to improving system reliability means that a similar component is connected with an original component in such a way that it is activated immediately upon failure of the original component. For this aspect of our analysis, the cold duplication method assumes that each component of a set C is improved by introducing a new but identical component with a perfect switch.

Let $c_i (i = 1,2, \dots, n)$ be the number of components in subsystem i, whose reliability is improved according to the cold duplication method with perfect switch, so $c_i \in \{0,1, \dots, m_i\}$ and $|C| = \sum_{i=1}^{n} c_i$.

Let $s_1(t)$ be the reliability function of each component whose reliability is improved according to cold duplication with perfect switch. Regarding a definition of cold duplication with perfect switch, we can describe this improvement as a renewal process with only one renewal (Gamiz et al., 2011). Using the convolution technique, the reliability function of each component whose reliability is improved according to cold duplication with perfect switch can be derived as:

$$s_1(t) = 1 - \int_0^t \frac{-dr(x)}{dx} [1 - r(t - x)] dx \tag{11}$$

where r(t) is the reliability function for the exponentiated Weibull lifetime distribution presented in Equation (1). By comparison with Equation (2), we see that the reliability function $R_i^{(C)}(t)$ of subsystem i is given by

$$R_i^{(C)}(t) = \{s_1(t)\}^{c_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - c_i} \tag{12}$$

for $t \geq 0$, from Equation (1), since the components are connected in series. Then the reliability function of the system takes the form

$$R_i^{(C)}(t) = 1 - \prod_{i=1}^{n} \left[1 - \{s_1(t)\}^{c_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - c_i}\right] \tag{13}$$

since the subsystems are connected in parallel. We can then compute the mean time to failure of this series-parallel system as

$$MTTF^{(C)} = \int_0^{\infty} \left(1 - \prod_{i=1}^{n} \left[1 - \{s_1(t)\}^{c_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - c_i}\right]\right) dt \tag{14}$$

3.2.3 *Cold duplication method with imperfect switch*

This approach to improving system reliability means that a similar component is connected with an original component by a cold standby via a random switch having a constant failure rate. For this aspect of our analysis, the cold duplication method assumes that each component of a set D is improved by introducing a new but identical component with an imperfect switch.

Let $d_i (i = 1,2, \dots, n)$ be the number of components in subsystem i, whose reliability is improved according to the cold duplication method with imperfect switch, so $d_i \in \{0,1, \dots, m_i\}$ and $|D| = \sum_{i=1}^{n} d_i$.

Let $s_2(t)$ be the reliability function of each component whose reliability is improved according to cold duplication with imperfect switch. Following the same technique that we used for cold duplication with perfect switch but with the extra condition that the switch is not 100% reliable (Billinton and Allan, 1992), we have

$$s_2(t) = 1 - \int_0^t \frac{-dr(x)}{dx}[1 - r(t-x)s_3(x)]dx \qquad (15)$$

where $r(\square)$ was defined previously for cold duplication with perfect switch, and $s_3(t)$ is the reliability function for the imperfect switch. The imperfect switch is chosen to have a constant failure rate λ, which means it has an exponential lifetime distribution with parameter λ

$$s_3(x) = e^{-\lambda x} \qquad (16)$$

The reliability function $R_i^{(D)}(t)$ of subsystem i, the reliability function of the series-parallel system $R^{(D)}(t)$, and the mean time to failure of this system $MTTF^{(D)}$ can be obtained by comparison with Equations (12,13,14).

4. Reliability equivalence factors

According to El-Damcese (2009), "A reliability equivalence factor is a factor by which a characteristic of components of a system design has to be multiplied in order to reach equality of a characteristic of this design and a different design regarded as a standard".

We compute two types of reliability equivalence measures. The first type involves survival reliability equivalence factors (SREFs) and these are determined from the reliability or survival function. The second type involves mean reliability equivalence factors (MREFs) and these are determined from the mean time to failure.

4.1 *Survival reliability equivalence factors*

The idea of SREFs is to assess what degrees of intervention are required to establish equivalence between the reliability functions of a system whose reliability is improved according to one of the duplication methods and a system whose reliability is improved according to the reduction method. That is, to derive the SREFs, we have to solve the following set of equations

$$R^{(A)}(t) = R^{(H)}(t) = \omega, \qquad H = B, C, D \qquad (17)$$

4.2 *Mean reliability equivalence factors*

The idea of MREFs is to assess what degrees of intervention are required to establish equivalence between the mean time to failure of a system whose reliability is improved according to one of the duplication methods and a system whose reliability is improved according to the reduction method. That is, to derive the MREFs, we have to solve the following set of equations

$$MTTF^{(A)}(t) = MTTF^{(H)}(t) = \omega, \qquad H = B, C, D \qquad (18)$$

Table 1 Hot survival reliability equivalence factors

	ω	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
$A_1^{(0,1)}$	0.1	0.7238	0.4111	-	-	-	-	-	-	-	-	-
	0.5	0.6009	-	-	-	-	-	-	-	-	-	-
	0.9	0.4519	-	-	-	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.1	0.8657	0.7330	0.6047	0.6482	0.6108	0.5591	0.4930	0.4250	0.4134	0.3944	0.3648
	0.5	0.8173	0.6203	0.4006	0.6483	0.5501	0.4239	0.2429	0.2666	0.1961	-	-
	0.9	0.7803	0.4800	-	0.6188	0.4345	-	-	-	-	-	-
$A_3^{(0,3)}$	0.1	0.9111	0.8251	0.7445	0.7714	0.7482	0.7167	0.6774	0.6384	0.6320	0.6216	0.6057
	0.5	0.8807	0.7603	0.6444	0.7767	0.7206	0.6554	0.5836	0.5910	0.5712	0.5444	0.5096
	0.9	0.8597	0.6998	0.5234	0.7675	0.6807	0.5790	0.4623	0.5035	0.4720	0.4312	0.3783
$A_1^{(1,0)}$	0.1	0.9182	0.8163	0.6981	0.7403	0.7042	0.6517	0.5804	0.5022	0.4884	0.4654	0.4290
	0.5	0.8111	0.5830	0.2579	0.6173	0.4929	0.3029	-	-	-	-	-
	0.9	0.7162	-	-	0.4671	-	-	-	-	-	-	-
$A_2^{(1,1)}$	0.1	0.9336	0.8459	0.7381	0.7773	0.7438	0.6943	0.6255	0.5487	0.5350	0.5122	0.4760
	0.5	0.8677	0.6963	0.4697	0.7226	0.6279	0.4953	0.2879	0.3159	0.2322	-	-
	0.9	0.8204	0.5318	-	0.6713	0.4839	-	-	-	-	-	-

$A_3^{(1,2)}$	0.1	0.9451	0.8730	0.7848	0.8167	0.7894	0.7491	0.6937	0.6327	0.6219	0.6041	0.5762
	0.5	0.9013	0.7773	0.6259	0.7959	0.7295	0.6419	0.5283	0.5410	0.5062	0.4552	0.3808
	0.9	0.8732	0.6922	0.3914	0.7749	0.6667	0.5078	0.1574	0.3384	0.2208	-	-
$A_4^{(1,3)}$	0.1	0.9537	0.8945	0.8248	0.8497	0.8284	0.7976	0.7565	0.7129	0.7055	0.6932	0.6744
	0.5	0.9222	0.8286	0.7224	0.8423	0.7940	0.7331	0.6600	0.6679	0.6467	0.6173	0.5780
	0.9	0.9030	0.7753	0.6084	0.8318	0.7587	0.6643	0.5433	0.5876	0.5539	0.5086	0.4473
$A_2^{(2,0)}$	0.1	0.9594	0.9095	0.8532	0.8731	0.8560	0.8315	0.7991	0.7647	0.7588	0.7491	0.7341
	0.5	0.9085	0.8090	0.7070	0.8230	0.7747	0.7169	0.6511	0.6580	0.6395	0.6141	0.5807
	0.9	0.8697	0.7185	0.5488	0.7828	0.7003	0.6026	0.4894	0.5295	0.4988	0.4590	0.4071
$A_3^{(2,1)}$	0.1	0.9634	0.9167	0.8617	0.8813	0.8645	0.8401	0.8073	0.7722	0.7661	0.7562	0.7407
	0.5	0.9235	0.8332	0.7333	0.8463	0.8004	0.7433	0.6757	0.6829	0.6635	0.6366	0.6009
	0.9	0.8954	0.7612	0.5929	0.8201	0.7441	0.6483	0.5297	0.5726	0.5399	0.4966	0.4390
$A_4^{(2,2)}$	0.1	0.9669	0.9239	0.8720	0.8907	0.8747	0.8512	0.8193	0.7846	0.7785	0.7685	0.7530
	0.5	0.9352	0.8563	0.7649	0.8679	0.8268	0.7742	0.7099	0.7169	0.6980	0.6715	0.6355
	0.9	0.9144	0.8008	0.6489	0.8513	0.7859	0.7004	0.5879	0.6296	0.5979	0.5548	0.4952
$A_5^{(2,3)}$	0.1	0.9700	0.9308	0.8831	0.9004	0.8856	0.8640	0.8344	0.8020	0.7963	0.7869	0.7723
	0.5	0.9443	0.8762	0.7968	0.8863	0.8507	0.8050	0.7486	0.7548	0.7381	0.7147	0.6826
	0.9	0.9283	0.8336	0.7071	0.8756	0.8211	0.7500	0.6559	0.6909	0.6644	0.6280	0.5771

5. Numerical results and analysis

Suppose that we have a series-parallel system consisting of two subsystems connected in parallel. The first subsystem has two components connected in series and the second subsystem has three components connected in series. This means that $n = 2, m_1 = 2, m_2 = 3$ and the total number of components is $m = 5$. All of the system's components are assumed to be independent and identically distributed, with lifetimes that behave according to an exponentiated Weibull distribution with parameters $\alpha = 1, \beta = 2, \theta = 3$. We define:

- $A_k^{(i,j)}, i = 0,1,2, j = 0,1,2,3$ and $k = i + j$ to represent a reduction method that requires us to reduce the failure rate of i components from the first subsystem and j from the second subsystem.
- $B_k^{(i,j)}, i = 0,1,2, j = 0,1,2,3$ and $k = i + j$ to represent hot duplication methods when i components are added to the first subsystem and j to the second subsystem.
- Same thing for $C_k^{(i,j)}$ and $D_k^{(i,j)}$ but for cold duplication with perfect and imperfect switch.

For this scenario, in Tables 1 the SREFs for hot duplication are calculated using Matlab according to the above formulae where ω is chosen to be 0.1, 0.5, 0.9. For more discussions based on the results presented in Table 1 and Figures 2 and 3, it may be observed that:

- Reducing the failure rate of one component in the second subsystem (which we denote as $A_1^{(0,1)}$) by setting $\rho = 0.7238$ improves the reliability of the system like adding one component to the second subsystem (which we denote as $B_1^{(0,1)}$) according to a hot duplication method where the reliability function of the system is chosen to be $\omega = 0.1$.
- Missing values of the SREFs mean that it is not possible to reduce the failure rate for the set A of components in order to improve the system reliability to be equivalent with the system reliability that can be obtained by improving the set B of components according to hot duplication methods. We can note the same thing clearly in Figure 3.
- Figure 2 explains the improvement strategies to calculate the SREFs.
- MTTFs increase with decreasing ρ for all possible sets A, see Figure 3.
- In the same manner, tables and figures for SREFs for cold duplication and MREFs for both hot and cold duplication con be derived and interpreted.

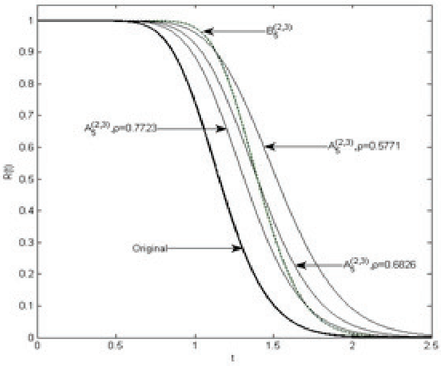

Figure 2 Survival reliability equivalence improvement strategies.

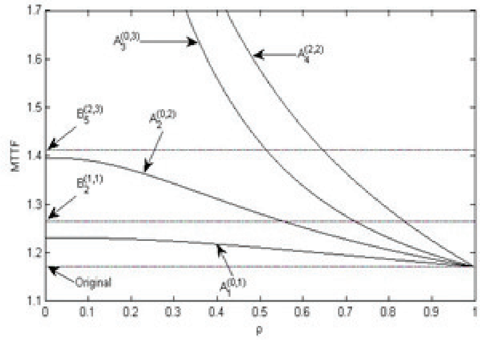

Figure 3 The behaviour of MTTF against $\boldsymbol{\rho}$.

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